Q/ Show that the sequence $\langle x_n \rangle$, where $x_n = \left(1+\frac{1}{n}\right)^n, \forall n \in N$ is convergent and it's limit lies between 2 and 3.

Sol$^n$: Given that

$$x_n = \left(1+\frac{1}{n}\right)^n, \forall n \in N, \text{ then using Binomial}$$

Theorem, we have

$$x_n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{\lfloor 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{1}{n}\right)^3$$
$$+ \cdots \frac{1}{\lfloor n} n(n-1)(n-2)\cdots(n-(n-1))\left(\frac{1}{n}\right)^n$$

$$\Rightarrow x_n = 1 + 1 + \frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right) + \frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$\cdots + \frac{1}{\lfloor n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) + \cdots$$
$$+ \cdots \qquad (1)$$

Again

$$x_{n+1} = 1 + 1 + \frac{1}{\lfloor 2}\left(1-\frac{1}{n+1}\right) + \frac{1}{\lfloor 3}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) +$$
$$\cdots + \frac{1}{\lfloor n+1}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \qquad (2)$$

Here, we know that $n+1 > n$

$$\Rightarrow \frac{1}{n+1} < \frac{1}{n}$$
$$\Rightarrow -\frac{1}{n+1} > -\frac{1}{n}$$
$$\Rightarrow \left(1-\frac{1}{n+1}\right) > \left(1-\frac{1}{n}\right). \qquad (3)$$

Similarly $\left(1-\frac{k}{n+1}\right) > \left(1-\frac{k}{n+1}\right) \qquad (4)$

using (3) & (4), we can see that

$$x_{n+1} > x_n \quad \forall n \in N$$

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$\Rightarrow \langle x_n \rangle$ is an increasing sequence.

Now Again from (1)

$$x_n = 1 + 1 + \frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right) + \frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots\cdots$$

$$\leq 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots + \frac{1}{\lfloor n} \cdots$$

$$\Rightarrow x_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

[This is because $\lfloor n = 1.2.3.\ldots.n$

$\Rightarrow \lfloor n > 1.2.2.2.2\ldots$ upto $(n-1)$ times.

$\Rightarrow \lfloor n > 2^{n-1}, \forall n \geq 2$

$\Rightarrow \frac{1}{\lfloor n} < \frac{1}{2^{n-1}}, \forall n \geq 2$ ]

$$\Rightarrow x_n < \frac{1\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}} = 1 + 2\left(1-\frac{1}{2^n}\right)$$

$\Rightarrow x_n < 3 - \frac{1}{2^{n-1}} \quad \forall n \in N$

$\Rightarrow x_n < 3 \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is bounded from above and it is also an increasing sequence.

$\Rightarrow \langle x_n \rangle$ is convergent.

It is clear from (1), that $x_n > 2$

$\Rightarrow 2 < x_n < 3$

$\Rightarrow 2 < \lim_{n\to\infty} x_n < 3$ H.P.

Q1 Prove that the sequence $\langle x_n \rangle$ is convergent.

$$x_1 > 0, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \forall n \in N.$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{1}{x_1}\right)$$

$$x_2 - x_1 = \frac{1}{2}\left(\frac{x_1^2+1}{x_1}\right) - x_1 = \frac{1}{2x_1}\left[1 - x_1^2\right]$$

$\Rightarrow x_2 - x_1 \geq 0$ if $x_1 \leq 1$

$\Rightarrow x_2 \geq x_1$ if $x_1 \leq 1$ ——— (i)

Similarly

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) - x_1$$

$$= \frac{1 - x_n^2}{2x_n} \geq 0 \text{ if } x_n \leq 1$$

$\Rightarrow x_{n+1} \geq x_n$ if $x_n \leq 1$

$\Rightarrow$ The sequence $\langle x_n \rangle$ is monotonically increasing if $x_1 \leq 1$ and decreasing if $x_n > 1$ $\forall n \in N$.

For both the cases it is monotonic and bounded.

$\Rightarrow$ It is convergent.

Let $\lim_{n\to\infty} x_n = l$

Now $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$

$$l = \frac{1}{2}\left(l + \frac{1}{l}\right)$$

$\Rightarrow 2l^2 = l^2 + 1 \Rightarrow l^2 = 1$

$\Rightarrow l = \pm 1$ But $l \neq -1 \because x_n > 0$

$\Rightarrow \lim_{n\to\infty} x_n = +1 \Rightarrow$ converges to 1

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